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Exact solutions of some nonlinear systems of partial differential equations by using the first integral method

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ABSTRACT

In recent years, many approaches have been utilized for finding the exact solutions of nonlinear systems of partial differential equations. In this paper, the first integral method introduced by Feng is adopted for solving some important nonlinear systems of partial differential equations, including, KdV, Kaup–Boussinesq and Wu–Zhang systems, analytically. By means of this method, some exact solutions for these systems of equations are formally obtained. The results obtained confirm that the proposed method is an efficient technique for analytic treatment of a wide variety of nonlinear systems of partial differential equations.

1. Introduction

With the availability of symbolic computation packages like Maple or Mathematica, direct searching for exact solutions of nonlinear systems of partial differential equations (PDEs) has become more and more attractive. Having exact solutions of nonlinear systems of PDEs makes it possible to study nonlinear physical phenomena thoroughly and facilitates testing the numerical solvers as well as aiding the stability analysis of solutions. In recent years, many approaches to nonlinear systems of PDEs have been employed, such as the extended tanh function method [1–4], the modified extended tanh function method [5,6], the exp-function method [7–9], the Weierstrass elliptic function method [10] and the first integral method [11]. Of these, the first integral method, which is based on the ring theory of commutative algebra, was first established by Feng [12–17]. The method has also been applied by many authors to solve different types of nonlinear equations in science and engineering. For example, the first integral method has been utilized to construct travelling wave solutions of the Cahn–Allen equation by Tascan and Bekir in [18]. Tascan et al. also have used this method [19] for constructing travelling wave solutions of nonlinear evolution equations. This method has also been adopted for solving some important nonlinear partial differential equations and the modified BBM equation, respectively in [20,21] to obtain new exact solutions for them. Recently, this method [22] has been successfully used for constructing the exact solutions of the nonlinear Schrödinger equation as well. Therefore, in the present article, the first integral method will be applied for analytic treatment of some important nonlinear systems of partial differential equations. The rest of this article has been arranged as follows: In Section 2, the basic ideas of the first integral method are expressed. In Section 3, the method is employed for obtaining the exact solutions of KdV, Kaup–Boussinesq and Wu–Zhang systems, and finally conclusions are presented in Section 4.

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2. First integral method

Step 1. Consider the following nonlinear system of partial differential equations with independent variables x and t and dependent variables u and v

$$F_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \ldots) = 0,$$
$$F_2(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \ldots) = 0. \tag{2.1}$$

Applying the transformations $u(x,t) = f(\varepsilon)$ and $v(x,t) = g(\varepsilon)$, where $\varepsilon = x - ct$, converts (2.1) into a system of ordinary differential equations (ODEs) as follows

$$G_1\big(f, g, f', g', \ldots\big) = 0,$$
$$G_2\big(f, g, f', g', \ldots\big) = 0, \tag{2.2}$$

where prime denotes the derivative with respect to the same variable ε.

Step 2. Using some mathematical operations, the system (2.2) is converted into a second-order ODE as

$$D\big(f, f', f''\big) = 0. \tag{2.3}$$

Step 3. By introducing new variables $X = f(\varepsilon)$ and $Y = f'$, (2.3) changes into a system of ODEs as the following form

$$\begin{cases} X' = Y, \\ Y' = H(X, Y). \end{cases} \tag{2.4}$$

Step 4. Now, the Division Theorem which is based on ring theory of commutative algebra, is adopted to obtain one first integral to (2.4), which reduces (2.3) to a first-order integrable ordinary differential equation. Finally, an exact solution to (2.1) is established, through solving the resulting first-order integrable differential equation.

Division Theorem. *Suppose that $P(x, y)$ and $Q(x, y)$ are polynomials of two variables x and y and $P(x, y)$ is irreducible in $\mathbb{C}[x, y]$. If $Q(x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial $T(x, y)$ in $\mathbb{C}[x, y]$ such that*

$$Q(x, y) = P(x, y)T(x, y).$$

The Divisor Theorem follows immediately from the Hilbert–Nullstellensatz Theorem [23].

Hilbert–Nullstellensatz Theorem. *Let K be a field and L be an algebraic closure of K. Then*:

(i) *Every ideal γ of $K[X_1, X_2, \ldots, X_n]$ not containing 1 admits at least one zero in L^n.*
(ii) *Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two elements of L^n. For the set of polynomials of $K[X_1, X_2, \ldots, X_n]$ zero at x to be identical with the set of polynomials of $K[X_1, X_2, \ldots, X_n]$ zero at y, it is necessary and sufficient that there exists a K-automorphism S of L such that $y_i = S(x_i)$ for $1 \leqslant i \leqslant n$.*
(iii) *For an ideal α of $K[X_1, X_2, \ldots, X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $K[X_1, X_2, \ldots, X_n]$ zero at x.*
(iv) *For a polynomial Q of $K[X_1, X_2, \ldots, X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $K[X_1, X_2, \ldots, X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.*

3. Applications

3.1. Exact solutions to the KdV system

Consider the following two-component evolutionary system of the homogeneous KdV equations [4]

$$\begin{cases} u_t - u_{xxx} - 2vu_x - uv_x = 0, \\ v_t - uu_x = 0. \end{cases} \tag{3.1.1}$$

By considering the transformations $u(x,t) = f(\varepsilon)$ and $v(x,t) = g(\varepsilon)$, and the wave variable $\varepsilon = x - ct$, (3.1.1) changes into a system of ordinary differential equations as follows

$$-cf' - f''' - 2gf' - fg' = 0, \tag{3.1.2}$$
$$-cg' - ff' = 0, \tag{3.1.3}$$

where prime denotes the derivative with respect to the same variable ε. We rewrite (3.1.3) as follows

$$g' + \frac{1}{c} f f' = 0. \tag{3.1.4}$$

Integration of (3.1.4), yields

$$g = \alpha - \frac{1}{2c} f^2, \tag{3.1.5}$$

where α is an integration constant. Now, substituting (3.1.5) into (3.1.2), gives

$$-(2\alpha + c) f' + \frac{2}{c} f^2 f' - f''' = 0.$$

We rewrite the equation above as

$$(2\alpha + c) f' - \frac{2}{c} f^2 f' + f''' = 0. \tag{3.1.6}$$

Integrating Eq. (3.1.6), now we will obtain

$$f'' = \beta - (2\alpha + c) f + \frac{2}{3c} f^3, \tag{3.1.7}$$

where β is an integrating constant. By introducing new variables $X = f(\varepsilon)$ and $Y = f'$, (3.1.7) changes into a system of ODEs

$$\begin{cases} X' = Y, \\ Y' = \beta - (2\alpha + c) X + \dfrac{2}{3c} X^3. \end{cases} \tag{3.1.8}$$

Now, the Division Theorem is employed to seek the first integral to (3.1.8). Suppose that $X = X(\varepsilon)$ and $Y = Y(\varepsilon)$ are the nontrivial solutions to (3.1.8), and $P(X, Y) = \sum_{i=0}^{m} a_i(X) Y^i$ is an irreducible polynomial in $\mathbb{C}[X, Y]$ such that

$$P\big(X(\varepsilon), Y(\varepsilon)\big) = \sum_{i=0}^{m} a_i(X) Y^i = 0, \tag{3.1.9}$$

$a_i(X)$, $i = 0, 1, 2, \ldots, m$ are polynomials of X, which $a_m(X) \neq 0$. Eq. (3.1.9) is also called the first integral to (3.1.8). According to the Division Theorem, there exists a polynomial $T(X, Y) = g(X) + h(X)Y$ in $\mathbb{C}[X, Y]$ so that

$$\frac{dP}{d\varepsilon} = \frac{\partial P}{\partial X}\frac{\partial X}{\partial \varepsilon} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \varepsilon} = \big(g(X) + h(X)Y\big)\left(\sum_{i=0}^{m} a_i(X) Y^i\right). \tag{3.1.10}$$

Case 1. Now assume that $m = 1$ in (3.1.9). By equating the coefficients of Y^i, $i = 0, 1, 2$ on both sides of (3.1.10), one has

$$a_1'(X) = h(X) a_1(X), \tag{3.1.11}$$

$$a_0'(X) = g(X) a_1(X) + h(X) a_0(X), \tag{3.1.12}$$

$$a_1(X)\left(\beta - (2\alpha + c)X + \frac{2}{3c} X^3\right) = g(X) a_0(X). \tag{3.1.13}$$

Since $a_i(X)$, $i = 0, 1$ are polynomials, then from (3.1.11) one can deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we consider $a_1(X) = 1$. Now, by balancing the degrees of $g(X)$ and $a_0(X)$, we can conclude that $\deg g(X) = 1$. Thus, by assuming that $g(X) = A_1 X + B_0$ such that $A_1 \neq 0$, from (3.1.12) one can obtain

$$a_0(X) = \frac{1}{2} A_1 X^2 + B_0 X + A_0,$$

where A_0 is an integration constant. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in (3.1.13) and equating the coefficient of each power of X to zero, we will derive a system of algebraic equations whose solution yields

$$A_0 = \frac{-2\alpha}{A_1} - \frac{4}{3A_1^3}, \qquad B_0 = 0, \qquad \beta = 0, \qquad c = \frac{4}{3A_1^2}. \tag{3.1.14}$$

Setting (3.1.14) in (3.1.9) leads to

$$Y+\left(\frac{1}{2}A_1X^2-\frac{2\alpha}{A_1}-\frac{4}{3A_1^3}\right)=0. \tag{3.1.15}$$

Now, by combining (3.1.15) and (3.1.8), a first-order ordinary differential equation is derived, which by solving this equation and considering $X=f(\varepsilon)$ and $u(x,t)=f(\varepsilon)$, we obtain

$$u(x,t)=\frac{2\sqrt{9\alpha A_1^2+6}}{3A_1^2}\tanh\left(\frac{\sqrt{9\alpha A_1^2+6}}{3A_1}\left(x-\frac{4}{3A_1^2}t+\varepsilon_0\right)\right),$$

where ε_0 is an arbitrary constant. Also, by considering the solution of the first-order differential equation and $X=f(\varepsilon)$ as well as the relations (3.1.5) and $v(x,t)=g(\varepsilon)$, we will obtain

$$v(x,t)=\alpha-\frac{3}{8}A_1^2\left[\frac{2\sqrt{9\alpha A_1^2+6}}{3A_1^2}\tanh\left(\frac{\sqrt{9\alpha A_1^2+6}}{3A_1}\left(x-\frac{4}{3A_1^2}t+\varepsilon_0\right)\right)\right]^2.$$

Case 2. It can be shown that when $m=2$ and $\deg g(X)=1$, the following exact solutions can be obtained

$$u_1(x,t)=\frac{4\sqrt{9\alpha A_1^2+24}}{3A_1^2}\tanh\left(\frac{\sqrt{9\alpha A_1^2+24}}{3A_1}\left(x-\frac{16}{3A_1^2}t+\varepsilon_0\right)\right),$$

$$v_1(x,t)=\alpha-\frac{3}{32}A_1^2\left[\frac{4\sqrt{9\alpha A_1^2+24}}{3A_1^2}\tanh\left(\frac{\sqrt{9\alpha A_1^2+24}}{3A_1}\left(x-\frac{16}{3A_1^2}t+\varepsilon_0\right)\right)\right]^2,$$

and

$$u_2(x,t)=\frac{4}{A_1(x-\frac{16}{3A_1^2}t)+\varepsilon_0},$$

$$v_2(x,t)=-\frac{8}{3A_1^2}-\frac{3}{2}A_1^2\left[\frac{1}{A_1(x-\frac{16}{3A_1^2}t)+\varepsilon_0}\right]^2,$$

where ε_0 is an arbitrary constant.

Remark 1. We have obtained three exact solutions to a two-component evolutionary system of the homogeneous KdV equations.

3.2. Exact solutions to the Kaup–Boussinesq system

Consider the Kaup–Boussinesq system [24]

$$\begin{cases} u_t-v_{xxx}-2vu_x-2uv_x=0, \\ v_t-u_x-2vv_x=0. \end{cases} \tag{3.2.1}$$

Applying the transformations $u(x,t)=f(\varepsilon)$ and $v(x,t)=g(\varepsilon)$, where $\varepsilon=x-ct$, converts (3.2.1) into a system of ordinary differential equations as

$$-cf'-g'''-2gf'-2fg'=0, \tag{3.2.2}$$

$$-cg'-f'-2gg'=0. \tag{3.2.3}$$

We can rewrite (3.2.3) in the form

$$f'+cg'+2gg'=0. \tag{3.2.4}$$

Integrating Eq. (3.2.4), we derive

$$f=\alpha-cg-g^2, \tag{3.2.5}$$

where α is an integrating constant. Now, inserting (3.2.5) into (3.2.2), yields

$$\left(c^2-2\alpha\right)g'+6cgg'+6g^2g'-g'''=0. \tag{3.2.6}$$

We rewrite Eq. (3.2.6) as

$$-\left(c^2-2\alpha\right)g'-6cgg'-6g^2g'+g'''=0. \tag{3.2.7}$$

Now, integration of (3.2.7), gives

$$g''=\beta+\left(c^2-2\alpha\right)g+3cg^2+2g^3, \tag{3.2.8}$$

where β is an integration constant. Introducing new variables $X=g(\varepsilon)$ and $Y=g'$, converts (3.2.8) into a system of ODEs

$$\begin{cases} X'=Y, \\ Y'=\beta+\left(c^2-2\alpha\right)X+3cX^2+2X^3. \end{cases} \tag{3.2.9}$$

According to the first integral method, assume that $X=X(\varepsilon)$ and $Y=Y(\varepsilon)$ are the nontrivial solutions to (3.2.9), and $P(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i$ is an irreducible polynomial in $\mathbb{C}[X,Y]$ such that

$$P\left(X(\varepsilon),Y(\varepsilon)\right)=\sum_{i=0}^{m}a_i(X)Y^i=0, \tag{3.2.10}$$

$a_i(X)$, $i=0,1,2,\ldots,m$ are polynomials of X, which $a_m(X)\neq 0$. Due to the Division Theorem, there exists a polynomial $T(X,Y)=g(X)+h(X)Y$ in $\mathbb{C}[X,Y]$ so that

$$\frac{dP}{d\varepsilon}=\frac{\partial P}{\partial X}\frac{\partial X}{\partial \varepsilon}+\frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \varepsilon}=\left(g(X)+h(X)Y\right)\left(\sum_{i=0}^{m}a_i(X)Y^i\right). \tag{3.2.11}$$

Now, suppose that $m=1$ in (3.2.10). By equating the coefficients of Y^i, $i=0,1,2$ on both sides of (3.2.11), one can obtain

$$a_1'(X)=h(X)a_1(X), \tag{3.2.12}$$

$$a_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{3.2.13}$$

$$a_1(X)\left(\beta+\left(c^2-2\alpha\right)X+3cX^2+2X^3\right)=g(X)a_0(X). \tag{3.2.14}$$

Since $a_i(X)$, $i=0,1$ are polynomials, then from (3.2.12) one can deduce that $a_1(X)$ is a constant and $h(X)=0$. For convenience, we consider $a_1(X)=1$. Now, by balancing the degrees of $g(X)$ and $a_0(X)$, we can conclude that $\deg g(X)=1$. Thus, by assuming that $g(X)=A_1X+B_0$ such that $A_1\neq 0$, from (3.2.13) we have

$$a_0(X)=\frac{1}{2}A_1X^2+B_0X+A_0,$$

where A_0 is an integrating constant. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in (3.2.14) and equating the coefficient of each power of X to zero, a system of algebraic equations can be obtained, which after being solved, we arrive at

$$A_1=2, \qquad c=\frac{-\beta}{\alpha}, \qquad A_0=-\alpha, \qquad B_0=\frac{-\beta}{\alpha}, \tag{3.2.15}$$

and

$$A_1=-2, \qquad c=\frac{-\beta}{\alpha}, \qquad A_0=\alpha, \qquad B_0=\frac{\beta}{\alpha}. \tag{3.2.16}$$

Setting (3.2.15) and (3.2.16) in (3.2.10) yields

$$Y+\left(X^2-\frac{\beta}{\alpha}X-\alpha\right)=0,$$

and

$$Y+\left(-X^2+\frac{\beta}{\alpha}X+\alpha\right)=0.$$

Now, by combining these equations with (3.2.9), two first-order ordinary differential equations are derived, which by solving these equations and considering $X=g(\varepsilon)$ and $v(x,t)=g(\varepsilon)$, we obtain

$$v_1(x,t)=\frac{1}{2\alpha}\left[\beta+\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right],$$
$$v_2(x,t)=\frac{1}{2\alpha}\left[\beta-\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right],$$

where ε_0 is an arbitrary constant. Also, by considering the solutions of two first-order differential equations and $X=g(\varepsilon)$ as well as the relations (3.2.5) and $u(x,t)=f(\varepsilon)$, we will obtain

$$\begin{aligned}u_1(x,t)&=\alpha+\frac{\beta}{\alpha}\left[\frac{1}{2\alpha}\left[\beta+\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right]\right]\\&\quad-\left[\frac{1}{2\alpha}\left[\beta+\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right]\right]^2,\\u_2(x,t)&=\alpha+\frac{\beta}{\alpha}\left[\frac{1}{2\alpha}\left[\beta-\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right]\right]\\&\quad-\left[\frac{1}{2\alpha}\left[\beta-\sqrt{4\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{4\alpha^3+\beta^2}}{2\alpha}\left(x+\frac{\beta}{\alpha}t+\varepsilon_0\right)\right)\right]\right]^2,\end{aligned}$$

where ε_0 is an arbitrary constant.

Remark 2. We have obtained two exact solutions to the Kaup–Boussinesq system.

3.3. Exact solutions to the Wu–Zhang system

Consider the Wu–Zhang system [25]

$$\begin{cases}u_t+uu_x+v_x=0,\\v_t+vu_x+uv_x+\frac{1}{3}u_{xxx}=0.\end{cases}\tag{3.3.1}$$

By considering the transformations $u(x,t)=f(\varepsilon)$ and $v(x,t)=g(\varepsilon)$, where $\varepsilon=x-ct$, (3.3.1) changes into a system of ODEs as

$$-cf'+ff'+g'=0,\tag{3.3.2}$$
$$-cg'+gf'+fg'+\frac{1}{3}f'''=0,\tag{3.3.3}$$

where prime denotes the derivative with respect to the same variable ε. Integrating Eq. (3.3.2), we will have

$$g=\alpha+cf-\frac{1}{2}f^2,\tag{3.3.4}$$

where α is an integration constant. Now, inserting (3.3.4) into (3.3.3), leads to

$$\left(-c^2+\alpha\right)f'+3cff'-\frac{3}{2}f^2f'+\frac{1}{3}f'''=0.\tag{3.3.5}$$

We rewrite Eq. (3.3.5) as follows

$$\left(-3c^2+3\alpha\right)f'+9cff'-\frac{9}{2}f^2f'+f'''=0.\tag{3.3.6}$$

Now, integration of (3.3.6), yields

$$f''=\beta+\left(3c^2-3\alpha\right)f-\frac{9}{2}cf^2+\frac{3}{2}f^3,\tag{3.3.7}$$

where β is an integrating constant. By introducing new variables $X=f(\varepsilon)$ and $Y=f'$, (3.3.7) changes into a system of ODEs as

$$\begin{cases}X'=Y,\\Y'=\beta+\left(3c^2-3\alpha\right)X-\frac{9}{2}cX^2+\frac{3}{2}X^3.\end{cases}\tag{3.3.8}$$

Now, the Division Theorem is employed to seek the first integral to (3.3.8). Suppose that $X = X(\varepsilon)$ and $Y = Y(\varepsilon)$ are the nontrivial solutions to (3.3.8), and $P(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in $\mathbb{C}[X,Y]$ such that

$$P\big(X(\varepsilon), Y(\varepsilon)\big) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{3.3.9}$$

$a_i(X)$, $i = 0, 1, 2, \dots, m$ are polynomials of X, which $a_m(X) \neq 0$. According to the Division Theorem, there exists a polynomial $T(X,Y) = g(X) + h(X)Y$ in $\mathbb{C}[X,Y]$ so that

$$\frac{dP}{d\varepsilon} = \frac{\partial P}{\partial X}\frac{\partial X}{\partial \varepsilon} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \varepsilon} = \big(g(X) + h(X)Y\big)\left(\sum_{i=0}^{m} a_i(X)Y^i\right). \tag{3.3.10}$$

Now, suppose that $m = 1$ in (3.3.9). By equating the coefficients of Y^i, $i = 0, 1, 2$ on both sides of (3.3.10), one can obtain

$$a_1'(X) = h(X)a_1(X), \tag{3.3.11}$$

$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X), \tag{3.3.12}$$

$$a_1(X)\left(\beta + \big(3c^2 - 3\alpha\big)X - \frac{9}{2}cX^2 + \frac{3}{2}X^3\right) = g(X)a_0(X). \tag{3.3.13}$$

Since $a_i(X)$, $i = 0, 1$ are polynomials, then from (3.3.11) one can deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Now, by balancing the degrees of $g(X)$ and $a_0(X)$, we can conclude that $\deg g(X) = 1$. Thus, by assuming that $g(X) = A_1X + B_0$ such that $A_1 \neq 0$, from (3.3.12) we have

$$a_0(X) = \frac{1}{2}A_1X^2 + B_0X + A_0,$$

where A_0 is an integrating constant. Setting $a_0(X)$, $a_1(X)$ and $g(X)$ in (3.3.13) and equating the coefficient of each power of X to zero, we will obtain a system of algebraic equations whose solution yields

$$A_1 = \sqrt{3}, \qquad c = \frac{\beta}{3\alpha}, \qquad A_0 = -\sqrt{3}\alpha, \qquad B_0 = \frac{-\beta}{\sqrt{3}\alpha}, \tag{3.3.14}$$

and

$$A_1 = -\sqrt{3}, \qquad c = \frac{\beta}{3\alpha}, \qquad A_0 = \sqrt{3}\alpha, \qquad B_0 = \frac{\beta}{\sqrt{3}\alpha}. \tag{3.3.15}$$

Substituting (3.3.14) and (3.3.15) into (3.3.9) gives

$$Y + \left(\frac{\sqrt{3}}{2}X^2 - \frac{\beta}{\sqrt{3}\alpha}X - \sqrt{3}\alpha\right) = 0,$$

and

$$Y + \left(\frac{-\sqrt{3}}{2}X^2 + \frac{\beta}{\sqrt{3}\alpha}X + \sqrt{3}\alpha\right) = 0,$$

respectively. In a manner similar to that described in previous subsections, one can obtain two exact solutions to the Wu–Zhang system as

$$u_1(x,t) = \frac{1}{3\alpha}\left[\beta + \sqrt{18\alpha^3 + \beta^2}\tanh\left(\frac{\sqrt{54\alpha^3 + 3\beta^2}}{6\alpha}\left(x - \frac{\beta}{3\alpha}t + \varepsilon_0\right)\right)\right],$$

$$\begin{aligned} v_1(x,t) = \alpha &+ \frac{\beta}{3\alpha}\left[\frac{1}{3\alpha}\left[\beta + \sqrt{18\alpha^3 + \beta^2}\tanh\left(\frac{\sqrt{54\alpha^3 + 3\beta^2}}{6\alpha}\left(x - \frac{\beta}{3\alpha}t + \varepsilon_0\right)\right)\right]\right] \\ &- \frac{1}{2}\left[\frac{1}{3\alpha}\left[\beta + \sqrt{18\alpha^3 + \beta^2}\tanh\left(\frac{\sqrt{54\alpha^3 + 3\beta^2}}{6\alpha}\left(x - \frac{\beta}{3\alpha}t + \varepsilon_0\right)\right)\right]\right]^2, \end{aligned}$$

and

$$u_2(x,t)=\frac{1}{3\alpha}\left[\beta-\sqrt{18\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{54\alpha^3+3\beta^2}}{6\alpha}\left(x-\frac{\beta}{3\alpha}t+\varepsilon_0\right)\right)\right],$$

$$v_2(x,t)=\alpha+\frac{\beta}{3\alpha}\left[\frac{1}{3\alpha}\left[\beta-\sqrt{18\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{54\alpha^3+3\beta^2}}{6\alpha}\left(x-\frac{\beta}{3\alpha}t+\varepsilon_0\right)\right)\right]\right]$$
$$-\frac{1}{2}\left[\frac{1}{3\alpha}\left[\beta-\sqrt{18\alpha^3+\beta^2}\tanh\left(\frac{\sqrt{54\alpha^3+3\beta^2}}{6\alpha}\left(x-\frac{\beta}{3\alpha}t+\varepsilon_0\right)\right)\right]\right]^2,$$

where ε_0 is an arbitrary constant.

Remark 3. We have obtained two exact solutions to the Wu–Zhang system.

Remark 4. All exact solutions were put back into the corresponding systems, by means of Maple software, and their satisfactions confirm the validity of the solutions obtained in this paper.

4. Conclusion

In this paper, the first integral method was successfully applied to solve some important nonlinear systems of partial differential equations, including, the KdV, Kaup–Boussinesq and Wu–Zhang systems analytically. By making use of this method, some exact solutions for these systems of equations were formally obtained. Due to the good performance of the first integral method, it is believed that this method is a promising technique in handling a wide variety of nonlinear systems of partial differential equations. Also, this method is computerizable, which allows us to perform complicated and tedious algebraic calculations on a computer.

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